

# Hausdorff Dimension, Lagrange and Markov Dynamical Spectra for Geometric Lorenz Attractors

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## Abstract

In this paper, we show that geometric Lorenz attractors have Hausdorff dimension strictly greater than 2. We use this result to show that for a “large” set of real functions the Lagrange and Markov Dynamical spectrum associated to these attractors has persistently non-empty interior.

## 1 Introduction

In 1963 the meteorologist E. Lorenz published in the Journal of Atmospheric Sciences [Lor63] an example of a parametrized polynomial system of differential equations

$$\begin{aligned} \dot{x} &= a(y - x) & a &= 10 \\ \dot{y} &= rx - y - xz & r &= 28 \\ \dot{z} &= xy - bz & b &= 8/3 \end{aligned} \tag{1}$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast.

Numerical simulations for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a stranger attractor, called the *Lorenz attractor*. However Lorenz’s equations proved to be very resistant to rigorous mathematical analysis, and also presented very serious difficulties to rigorous numerical study.

A very successful approach was taken by Afraimovich, Bykov and Shil’nikov [ABS77], [ABS82], and Guckenheimer, Williams [GW79], independently: they constructed the so called *geometric Lorenz models* for the behavior observed by Lorenz, (see Section 2) for precise definition. These models are flows in 3-dimensions for which one can rigorously prove the coexistence of an equilibrium point accumulated by regular orbits. Recall that a regular solution is an orbit where the flow does not vanish. Most remarkably, this

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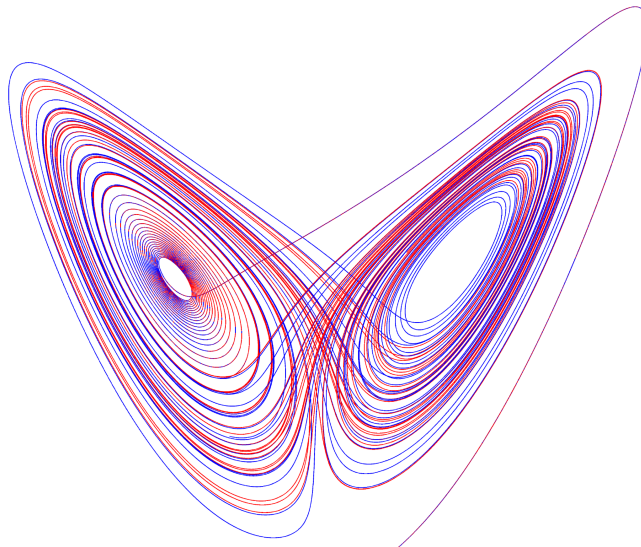


Figure 1: Lorenz attractor

attractor is robust: it can not be destroyed by small perturbation of the original flow. Taking into account that the divergence of the vector field induced by the system (1) is negative, it follows that the Lebesgue measure of the Lorenz attractor is zero. Henceforth, it is natural to ask about its Hausdorff dimension. Numerical experiments give that this value is approximately equal to 2.062 (cf. [Vis04]) and also, for some parameter, the dimension of the physical invariant measure is lies in the interval  $[1.24063, 1.24129]$  (cf. [GN16]). In this paper we address the problem to prove that the Hausdorff dimension of a geometric Lorenz attractor is strictly greater than 2. In [AP83] and [Ste00], this dimension is characterized in terms of the pressure of the system and in terms of the Lyapunov exponents and the entropy with respect to a good invariant measure associated to the geometric model. But, in both cases, the authors prove that the Hausdorff dimension is greater or equal than 2, but not necessary strictly greater than 2. A first attempt to obtain the strict inequality was given in [ML08], where the authors achieve this result in the particular case that both branches of the unstable manifold of the equilibrium meet the stable manifold of the equilibrium. But this condition is quite strong and extremely unstable. Our goal in this paper is to recovery the proof of the strict inequality for the Hausdorff dimension for any geometric Lorenz attractor, proving the following result:

**Theorem A.** *Geometric Lorenz attractors have Hausdorff dimension strictly greater than two.*

In order to achieve our goal, we first study the discrete time system associated to geometric Lorenz model. We prove that the Hausdorff dimension of the attractor restricted to a cross-section is greater than 1 and as a consequence, we get that the Hausdorff dimension of the attractor for the flow is strictly greater than 2. To announce our first result, recall that a geometric Lorenz attractor  $\Lambda$  admits a global cross-section  $S$  and a partial hyperbolic first return map  $P : S \rightarrow S$  preserving a stable foliation  $\mathcal{F}$ . Let  $f$  be the quotient map defined on space of leaves and denote by  $\Lambda_P$  the attractor for  $P$ . Theorem A is an immediate consequence of the following result:

**Theorem B.** *The Hausdorff dimension of  $\Lambda_P$  is strictly greater than 1.*

The idea of the proof of Theorem B is to construct an increasing family of Cantor sets for the quotient map  $f$  with Hausdorff dimension very close to 1. And to achieve this, we shall use the existence of an absolutely continuous invariant measure with respect to Lebesgue measure for the map  $f$ .

To announce the next result, let us recall the classical notions of Lagrange and Markov spectra. The *Lagrange spectrum*  $\mathcal{L}$  is a classical subset of the extended real line, related to Diophantine approximations and can be described as the set

$$\mathcal{L} = \left\{ k(\alpha) := \limsup_{p,q \rightarrow \infty} |q(q\alpha - p)|^{-1} ; \alpha \in \mathbb{R} \right\} \subset \mathbb{R} \cup \{+\infty\}.$$

In other words,  $k \in \mathcal{L}$  if and only if there exists  $\alpha \in \mathbb{R}$  such that, for any  $c > k$ , we have  $|\alpha - \frac{p}{q}| > \frac{1}{cq^2}$  for all  $p$  and  $q$  big enough and, moreover,  $k$  is minimal with respect to this property. Another interesting set related to Diophantine approximations is the classical *Markov spectrum* defined by

$$\mathcal{M} = \left\{ \inf_{(x,y) \in \mathbb{Z}^2 \setminus (0,0)} |f(x,y)|^{-1} : f(x,y) = ax^2 + bxy + cy^2 \text{ with } b^2 - 4ac = 1 \right\}.$$

The structure of  $\mathcal{L}$  and  $\mathcal{M}$  has been studied for more than a century, from the works of Markoff (1879), Hurwitz (1891) and Hall (1947). Recently, Moreira [Mor02] has new results on this. For more on Lagrange and Markov spectra we refer to the interested reader the book by Cusick and Flahive, [CF89].

One can show that  $\mathcal{L}$  can also be described as a penetration spectrum for the geodesic flow on the unit tangent bundle  $SN$  of the modular surface  $N = \mathbb{H}/SL(2, \mathbb{Z})$  in the following way. If  $\gamma(t)$  is any hyperbolic geodesic on  $N$  which has  $\alpha$  as forward endpoint, the value  $k(\alpha)$  is related to the geometric quantity

$$\limsup_{t \rightarrow \infty} \text{height}(\gamma(t))$$

where  $\text{height}(\cdot)$  denotes the hyperbolic height. This quantity gives the asymptotic depth of penetration of the ray  $\gamma(t)$  into the cusp of the modular surface.

Several generalizations of the classical Lagrange spectrum have been studied by many authors, in particular, in the context of Fuchsian groups and, more in general, negatively curved manifolds see [Ser85, Ser88, HS86, Vul97, Vul00, Vul95, HMU12, AMU14, PP10, PP09, PP11, HP10, RM15b]. For a very brief survey of these generalizations, we refer to the introduction of [Rom16] and [RM15b]. In the context of hyperbolic dynamics, we refer to [RM15a] and [Rom16], where it was proved that for typical horseshoes (with Hausdorff dimension greater than 1) and conservative Anosov flows, the Lagrange and Markov dynamical spectrum has non-empty interior for typical functions. In this article we prove these last results for any geometric Lorenz attractor.

Let  $X$  be a vector field that defines the geometric Lorenz attractor  $\Lambda$  and  $U$  an open neighborhood of  $\Lambda$  where  $X$  is defined. If  $f \in C^0(U, \mathbb{R})$  then the *Lagrange Dynamical*

*Spectrum* associated to  $(f, \Lambda)$  is defined by

$$L(f, \Lambda) = \left\{ \sup_{t \in \mathbb{R}} f(X^t(x)) : x \in \Lambda \right\}$$

and the *Markov Dynamical Spectrum* associated to  $(f, \Lambda)$  is defined by

$$M(f, \Lambda) = \left\{ \sup_{t \in \mathbb{R}} f(X^t(x)) : x \in \Lambda \right\}.$$

In this context, the second result of this paper is:

**Theorem C.** *Let  $\Lambda$  be a geometric Lorenz attractor associated to a flow  $X^t$ . Then arbitrarily close to  $X^t$ , there are a flow  $X_0^t$  and a neighborhood  $\mathcal{W}$  of  $X_0^t$  such that, if  $\Lambda_Y$  denotes the geometric Lorenz attractor associated to  $Y \in \mathcal{W}$ , there is an open and dense set  $\mathcal{H}_Y \subset C^1(U, \mathbb{R})$  such that for all  $f \in \mathcal{H}_Y$ , we have*

$$\text{int } L(f, \Lambda_Y) \neq \emptyset \text{ and } \text{int } M(f, \Lambda_Y) \neq \emptyset,$$

where  $\text{int } A$  denotes the interior of  $A$ .

## 2 Geometrical Model

In this section we present, for completeness, a detailed construction of the geometric Lorenz attractor. We follow near [GP10, AP10]. For this, we first analyze the dynamics in a neighborhood of the singularity, *i.e.*, at the origin, and then we imitate the effect of the pair of saddle singularities in the original Lorenz flow.

### 2.1 Near the singularity

By the Hartman-Grobman Theorem or by the results of Sternberg [Ste58], in a neighborhood of the origin the Lorenz equations are equivalent to the linear system  $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$  through conjugation, thus

$$X^t(x_0, y_0, z_0) = (x_0 e^{\lambda_1 t}, y_0 e^{\lambda_2 t}, z_0 e^{\lambda_3 t}),$$

where  $\lambda_1 \approx 11.83$ ,  $\lambda_2 \approx 22.83$ ,  $\lambda_3 = \frac{8}{3}$  and  $(x_0, y_0, z_0) \in \mathbb{R}^3$  is an arbitrary initial point near  $(0, 0, 0)$ .

Consider  $S = \{(x, y, 1) : |x| \leq 1/2, |y| \leq 1/2\}$  and

$$S^- = \{(x, y, 1) : x < 0\}, \quad S^+ = \{(x, y, 1) : x > 0\} \text{ and}$$

$$S^* = S^- \cup S^+ = S \setminus \Gamma, \quad \text{where } \Gamma = \{(x, y, 1) : x = 0\}. \quad (2)$$

Assume that  $S$  is a transverse section to the flow so that every trajectory eventually crosses  $S$  in the direction of the negative  $z$  axis as in Figure 2. Consider also  $\Sigma = \{(x, y, z) : |x| = 1\} = \Sigma^- \cup \Sigma^+$  with  $\Sigma^\pm = \{(x, y, z) : x = \pm 1\}$ . For each  $(x_0, y_0, 1) \in S^*$  the time  $\tau$  such that  $X^\tau(x_0, y_0, 1) \in \Sigma$  is given by  $\tau(x_0, y_0, z_0) = -\frac{1}{\lambda_1} \log |x_0|$ , which only depends on

$x_0$  and is such that  $\tau(x_0) \rightarrow +\infty$  when  $x_0 \rightarrow 0$ . This is one of the reasons many standard numerical algorithms were unsuited to tackle the Lorenz system of equations. Hence we get (where  $\text{sgn}(x) = x/|x|$  for  $x \neq 0$  as usual)

$$X^\tau(x_0, y_0, 1) = (\text{sgn}(x_0), y_0 e^{\lambda_2 \tau(x_0)}, e^{\lambda_3 \tau(x_0)}) = (\text{sgn}(x_0), y_0 |x_0|^{-\frac{\lambda_2}{\lambda_1}}, |x_0|^{-\frac{\lambda_3}{\lambda_1}}).$$

Since  $0 < -\lambda_3 < \lambda_1 < -\lambda_2$ , we have  $0 < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$ . Let  $L: S^* \rightarrow \Sigma$  be

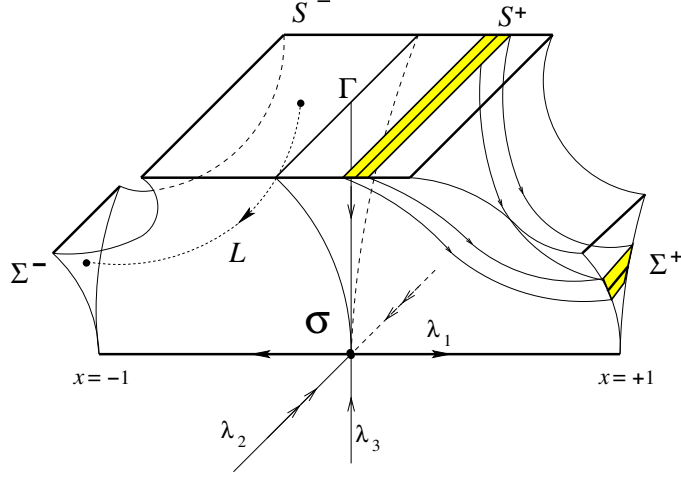


Figure 2: Behavior near the origin

given by

$$L(x, y, 1) = (\text{sgn}(x), y|x|^\beta, |x|^\alpha). \quad (3)$$

It is easy to see that  $L(S^\pm)$  has the shape of a triangle without the vertex  $(\pm 1, 0, 0)$ . In fact  $(\pm 1, 0, 0)$  are cusp points of the boundary of each of these sets.

From now on we denote by  $\Sigma^\pm$  the closure of  $L(S^\pm)$ . Clearly each line segment  $S^* \cap \{x = x_0\}$  is taken to another line segment  $\Sigma \cap \{z = z_0\}$  as sketched in Figure 2.

## 2.2 The effect of saddles.

To imitate the random turns of a regular orbit around the origin and obtain a butterfly shape for the flow, as in the original Lorenz flow (see Figures 1 and 2), we proceed as follows. The sets  $\Sigma^\pm$  should return to the cross section  $S$  through a flow described by a suitable composition of a rotation  $R^\pm$ , an expansion  $E_{\pm\theta}$  and a translation  $T_\pm$ . We are assuming that the “cusp triangles”  $L(S^\pm)$  are compressed in the  $y$ -direction and stretched on the other transverse direction and that this return map takes line segments  $\Sigma \cap \{z = z_0\}$  into line segments  $S \cap \{x = x_1\}$ , as sketched in Figure 3.

The rotation  $R_\pm$  has axis parallel to the  $y$ -direction, which is orthogonal to the  $x$ -direction. More precisely, if  $(x, y, z) \in \Sigma^\pm$ , then

$$R_\pm = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}.$$

The expansion occurs only along the  $x$ -direction, and so  $E_{\pm\theta}$  is given for

$$E_{\pm\theta} = \begin{pmatrix} \pm\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with  $\theta.2^{-\alpha} < 1$  and  $\theta.\alpha.2^{1-\alpha} > 1$ .

The translations  $T_{\pm}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  are chosen such that the unstable direction starting from the origin is sent to the boundary of  $S$  and the images of both  $\Sigma^{\pm}$  are disjoint, for more details see [AP10, Chapter 3, Section 3] and [GP10].

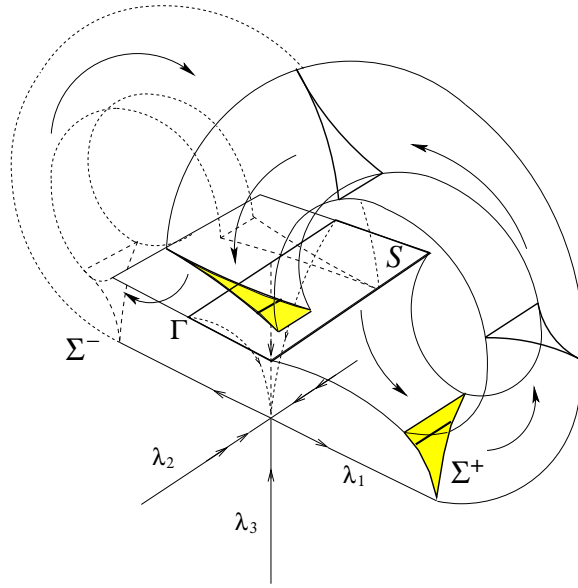


Figure 3:  $P$  takes  $\Sigma^{\pm}$  to  $S$ .

These transformations  $R_{\pm}$ ,  $E_{\pm\theta}$  and  $T_{\pm}$  take line segments  $\Sigma^{\pm} \cap \{z = z_0\}$  into line segments  $S \cap \{x = x_1\}$  as shown in Figure 3, and so does the composition  $T_{\pm} \circ E_{\pm\theta} \circ R_{\pm}$ .

This composition of linear maps describes a vector field in a region outside  $[-1, 1]^3$ , in the sense that one can use the above linear maps to define a vector field  $X$  such that the first return map to  $S$  of the associated flow realizes  $T_{\pm} \circ E_{\pm\theta} \circ R_{\pm}$  as a map  $\Sigma^{\pm} \rightarrow S$ . We note that the flow on the attractor we are constructing will pass through the region between  $\Sigma^{\pm}$  and  $S$  a relatively small time with respect to the linearized region. The linearized regions will then dominate all estimates of expansion/contraction.

The above construction enables us to describe, for  $t \in \mathbb{R}^+$ , the orbit  $X^t(x)$  of each point  $x \in S$ : the orbit will start following the linear field until  $\Sigma^{\pm}$  and then it will follow  $X$  coming back to  $S$  and so on. Let us denote by  $W = \{X^t(x) : x \in \Sigma; t \in \mathbb{R}^+\}$  the set where this flow acts. The geometric Lorenz flow is the couple  $(W, X^t)$  and putting  $\Gamma = \{(x, y, 1) : x = 0\} \subset S$ , the geometric Lorenz attractor is the set

$$\Lambda = \overline{\bigcap_{t \geq 0} X^t(\Lambda_P)}, \text{ where } \Lambda_P = \overline{\bigcap_{i \geq 1} P^i(S \setminus \Gamma)}. \quad (4)$$

### 2.3 The first return map.

Let  $S^* = S \setminus \Gamma$ . The Poincaré first return map  $P: S^* \rightarrow S$  is defined by

$$P(x, y) = \begin{cases} T_+ \circ E_{+\theta} \circ R_+ \circ L(x, y, 1) & \text{for } x > 0 \\ T_- \circ E_{-\theta} \circ R_- \circ L(x, y, 1) & \text{for } x < 0 \end{cases} . \quad (5)$$

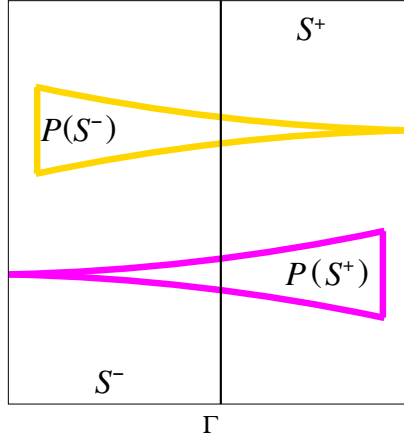


Figure 4: The return map image  $P(S^*)$

The above combined effects imply that the foliation of  $S$  given by the lines  $S \cap \{x = x_0\}$  is invariant under Poincaré first return map, meaning that for any given leaf  $\gamma$  of this foliation, its image  $P(\gamma)$  is contained in a leaf of the same foliation. Hence  $P$  must have the form  $P(x, y) = (f(x), g(x, y))$  for some functions  $f: I \setminus \{0\} \rightarrow I$  and  $g: (I \setminus \{0\}) \times I \rightarrow I$ , where  $I = [-1/2, 1/2]$ .

Combining the definition of  $L$  from we see that

$$f(x) = \begin{cases} f_0(x^\alpha) & \text{if } x > 0 \\ f_1(x^\alpha) & \text{if } x < 0 \end{cases} ; \quad \text{with } f_i = (-1)^i \theta \cdot x + b_i \quad i = 1, 2;$$

and

$$g(x, y) = \begin{cases} g_0(x^\alpha, y \cdot x^\beta) & \text{if } x > 0 \\ g_1(x^\alpha, y \cdot x^\beta) & \text{if } x < 0 \end{cases} .$$

where  $g_1: I^- \times I \rightarrow I$  and  $g_0: I^+ \times I \rightarrow I$  are suitable affine maps, with  $I^- = [-1/2, 0)$  and  $I^+ = (0, 1/2]$ .

### 2.4 Properties of the one-dimensional map $f$ .

Here we specify the properties of the one-dimensional map  $f$  described above:

- (f1) the symmetry of the Lorenz equations implies  $f(-x) = -f(x)$ . This is not essential in what follows and is not preserved under perturbation of the flow;
- (f2)  $f$  is discontinuous at  $x = 0$  with lateral limits  $f(0^-) = \frac{1}{2}$  and  $f(0^+) = -\frac{1}{2}$ , since  $P$  is not defined at  $\Gamma$  because  $\Gamma \subset W^s(0, 0, 0)$ ;

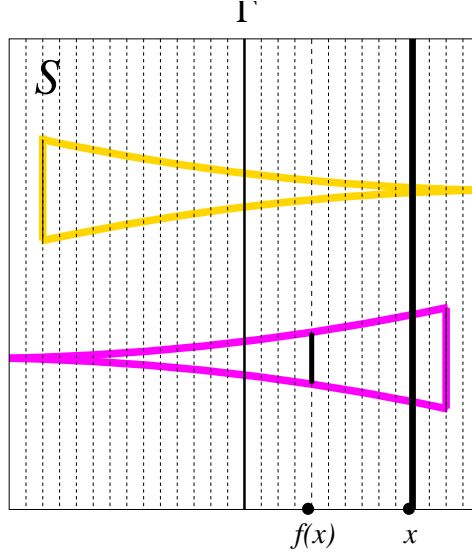


Figure 5: Projection on  $I$  through the stable leaves and a sketch of the image of one leaf under the return map.

(f3)  $f$  is differentiable on  $I \setminus \{0\}$  and  $f'(x) > \sqrt{2}$ ;

(f4) the lateral limits of  $f'$  at  $x = 0$  are  $f'(0^-) = +\infty$  and  $f'(0^+) = -\infty$ .

We note that  $g: S^* \rightarrow I$  is defined in such a way that it contracts the second coordinate:  $g'_y(w) \leq \mu < 1$  for all  $w \in S^*$ , and the rate of contraction of  $g$  on the second coordinate should be much higher than the expansion rate of  $f$ , (Figure 4 sketches  $P(S^*)$ ). In addition the expansion rate is big enough to obtain a strong mixing property for  $f$  (namely, it is locally eventually onto, see Section 2.8).

## 2.5 Properties of the map $g$ .

By definition  $g$  is piecewise  $C^2$  and the following bounds on its partial derivatives hold:

(a) For all  $(x, y) \in S^*$  with  $x \neq 0$ , we have  $|\partial_y g(x, y)| = |x|^\beta$ . As  $\beta > 1$  and  $|x| \leq 1/2$  there is  $0 < \lambda < 1$  such that

$$|\partial_y g| < \lambda.$$

(b) For  $(x, y) \in S^*$  with  $x \neq 0$ , we have  $\partial_x g(x, y) = \beta x^{\beta-\alpha}$ . Since  $\beta > \alpha$  and  $|x| \leq 1/2$  we get  $|\partial_x g| < \infty$ .

Note that item (a) above implies first that there is a very strong domination of the contraction along the  $y$ -direction over the expansion along the  $x$ -direction, that is,

$$\frac{|\partial_y g(x, y)|}{|Df(x)|} \approx |x|^{\beta-\alpha+1} \approx \frac{|x|^\beta}{|Df(x)|} \quad \text{with } \beta - \alpha + 1 > 1.$$

Second, it implies that the foliation  $\mathcal{F}_X$  of  $S$  whose leaves are the lines  $S \cap \{x = \text{constant}\}$  is uniformly contracted by  $P$ , i. e., there is a constant  $C > 0$  such that, for any given leaf  $\gamma$  of  $\mathcal{F}_X$  and for  $y_1, y_2 \in \gamma$ , then

$$\text{dist}(P^n(y_1), P^n(y_2)) \leq C\lambda^n \text{dist}(y_1, y_2) \quad \text{when } n \rightarrow \infty.$$



Thus, the study of the maximal invariant set  $\Lambda$  inside the trapping region

$$U := \{X^t(x, y, 1) : (x, y, 1) \in S, 0 \leq t \leq \tau_X(x, y)\} \cup \{(0, 0, 0)\}$$

for this 3-flow can be reduced to the study of a bi-dimensional map, where  $\tau_X$  is the first return time of the orbit of  $(x, y, 1) \in S$  under  $X^t$  to  $S$ . Moreover, the dynamics of this map can be further reduced to a one-dimensional map, because the invariant contracting foliation  $\mathcal{F}_X$  enables us to identify two points at the same leaf, since their orbits remain forever on the same leaf and the distance of their images tends to zero under iteration. See Figure 5 for a sketch of this identification.

The quotient map  $f: S^*/\mathcal{F}_X \rightarrow S/\mathcal{F}_X$  obtained through the identification  $\pi: S \rightarrow S/\mathcal{F}_X$  is called the (one-dimensional) *Lorenz map*. It satisfies  $f \circ \pi = \pi \circ P$  by construction. Figure 6 shows the graph of this one-dimensional map.

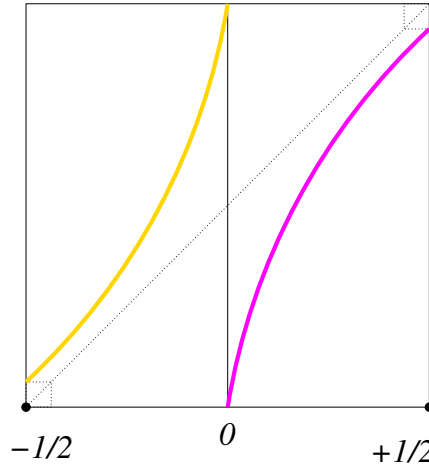


Figure 6: The Lorenz map

## 2.6 Persistence and smoothness of the contracting foliation.

The following persistence property is a consequence of the domination of the contraction along the  $y$ -direction over the expansion along the  $x$ -direction (see e.g. [AP10]).

**Theorem 2.1.** *Let  $X$  be the vector field obtained in the construction of the geometric Lorenz model and  $\mathcal{F}_X$  the invariant contracting foliation of the cross-section  $S$ . Then any vector field  $Y$  which is sufficiently  $C^1$ -close to  $X$  admits an invariant contracting continuous foliation  $\mathcal{F}_Y$  on the cross-section  $S$  with  $C^1$  leaves.*

Thus, the geometric Lorenz attractor constructed in the previous section is robust, that is, it persists for all nearby vector fields. More precisely: there exists a neighborhood  $U$  in  $R^3$  containing the attracting set  $\Lambda$ . such that for all vector fields  $Y$  which are  $C^1$ -close to  $X$  the maximal invariant subset in  $U$ , that is  $\Lambda_Y = \bigcap_{t \geq 0} Y^t(U)$ , is still a transitive  $Y$ -invariant set.

We note that  $\mathcal{F}_Y$  is a continuous foliation with  $C^1$  leaves. It can be shown that the holonomies along the leaves are in fact Hölder- $C^1$  (see [AP10]). Moreover, if we have a

strong dissipative condition on the equilibrium  $O$ , that is, if  $\beta > \alpha + k$  for some  $k \in \mathbb{Z}^+$  (see the definitions of  $\alpha, \beta$  as functions of the eigenvalues of 0 in (3)), it can be show that  $\mathcal{F}_Y$  is a  $C^k$  smooth foliation [SV16], and so the holonomies along its leaves are  $C^k$  maps. In particular, for strongly dissipative Lorenz attractors with  $\beta > \alpha + k$  the one-dimensional quotient map is  $C^k$  smooth away from the singularity (cf. [SV16]).

## 2.7 Robustness of geometric Lorenz attractors.

For  $Y$  sufficiently  $C^1$ -near  $X$ , let  $f_Y$  be the quotient map  ${}_Y: S^*/\mathcal{F}_Y \rightarrow S/\mathcal{F}_Y$  associated to the corresponding Poincaré map  $P_Y$ . Since the leaves of  $\mathcal{F}_Y$  are  $C^1$  close to those of  $\mathcal{F}_X$ , it follows that  $f_Y$  is  $C^1$  close to  $f$  and thus there exists  $c_Y \in [-1/2, 1/2]$  which plays, for  $f_Y$ , the same role as the singular point at 0. Thus, after a linear change of coordinates, we can assume that  $c_Y = 0$ , for all  $Y$ , and properties (f2)-(f4) from subsection 2.4 are still valid, albeit with different constants, for  $f_Y$  on a subinterval  $[-b_0, b_1]$  for some  $0 < b_0, b_1 < 1/2$  close to  $1/2$ . In particular

$$Df_Y(x) \approx |x|^{\alpha-1}, \quad i.e., \quad \frac{1}{C} \leq \frac{Df_Y(x)}{|x|^{\alpha-1}} \leq C \quad \text{and} \quad \frac{|Df_Y^2(x)|}{|x|^{\alpha-2}} \leq C_1 \quad (6)$$

for some  $C, C_1 > 1$  uniformly on a  $C^2$  neighborhood of  $X$ , where  $\alpha = \alpha(Y) = -\frac{\lambda_3(Y)}{\lambda_1(Y)}$  depends smoothly on  $Y$ . Finally, the condition (f3) ensures that  $f_Y$  has enough expansion to easily prove that every  $f_Y$  is *locally eventually onto* for all  $Y$  close to  $X$ , *i.e.*, for any interval  $J \subset (-b_0, b_1)$  there exists an iterate  $n \geq 1$  such that  $f_Y^n(J) = (-b_0, b_1)$  (see Lemma 2.1). By another change of coordinates, there is no loss of generality assume that both  $b_0, b_1$  are equal to  $1/2$  in what follows.

## 2.8 Topological and Ergodic Properties of one-dimensional map

Properties (f2) - (f4) imply another very important features for the one-dimensional map. Let us describe the ones we shall use in the sequel (cf. [Wil79]).

**Lemma 2.1.** [AP10, Lemma 3.16] *Let  $f: [-1/2, 1/2] \setminus \{0\} \rightarrow [-1/2, 1/2]$  satisfying properties (f2) - (f4) at section 2.4. Then  $f$  is locally eventually onto: for any open interval  $J$  not containing 0 there exists  $n$  such that  $f^n|_J$  is a diffeomorphism between  $J$  and one of the intervals  $(-1/2, 0)$  or  $(0, 1/2)$  (and the next iterate covers the interval  $(f(-1/2), f(1/2))$ ).*

*Proof.* Let  $J_0 \subset (-1/2, 1/2)$  be an open interval with  $0 \notin J_0$  and let  $\eta = \inf |f'| > \sqrt{2}$ . Since  $0 \notin J_0$  then  $f(J_0)$  is such that  $\ell(f(J_0)) \geq \eta \cdot \ell(J_0)$ , where  $\ell(\cdot)$  denotes length, and  $f(J_0)$  is connected.

1. If  $0 \notin f(J_0)$ , setting  $J_1 = f^2(J_0)$  we get  $\ell(J_1) \geq \eta^2 \ell(J_0)$ .
2. If  $0 \in f(J_0)$ , then  $f^2(J_0) = J^- \cup J^+$  where  $J^+$  is the biggest connected component.

Thus

$$\ell(J^+) \geq \frac{\ell(f^2(J_0))}{2} \geq \frac{\eta^2}{2} \ell(J_0).$$

Now replace  $J_0$  by  $J^+$  in case (2) or by  $J_1$  in case (1). Since  $\min\{\eta, \frac{\eta^2}{2}\} > 1$  we obtain after finitely many steps one of the intervals  $(-1/2, 0)$  or  $(0, 1/2)$ . One more iterate covers the interval  $(f(-1/2), f(1/2))$ . This concludes the proof.  $\square$

The next Lemma gives us the following ergodic property ([Via97, Corollary 3.4]).

**Lemma 2.2.** *Let  $f: [-1/2, 1/2] \setminus \{0\} \rightarrow [-1/2, 1/2]$  satisfying properties (f2) - (f4) at section 2.4. Then  $f$  has some absolutely continuous invariant probability measure (with respect to Lebesgue measure  $m$ ). Moreover, if  $\mu$  is any such measure then  $\mu = \varphi m$  where  $\varphi$  has bounded variation.*

### 3 Fat Cantor sets for $f$

The main goal in this section is to prove that there are regular Cantor sets for the map  $f$ , with Hausdorff dimension ( $HD$ ) very close to 1. More specifically, we prove

**Theorem 1.** *There is an increasing family of regular Cantor sets for  $f$ , say  $C_k$ , so that*

$$HD(C_k) \rightarrow 1 \quad \text{as } k \rightarrow +\infty.$$

*Proof.* We construct the Cantor sets inductively. Denote  $L_1 = [-1/2, 0)$  and  $L_2 = (0, 1/2]$ , and pick any interval  $I_1 \subset L_1$ . Lemma 2.1 implies that there is an iterate  $f^{n_1}$  of  $f$  such that  $f^{n_1}: I_1 \rightarrow L_1$  is a diffeomorphism. Let  $\{J_1^1, J_2^1\}$  be the complementary intervals in  $L_1$  of  $I_1$ . Again Lemma 2.1 implies that there are  $I_1^1 \subset J_1^1$ ,  $I_2^1 \subset J_2^1$ ,  $n_1^1$  and  $n_2^1$  such that  $f^{n_i^1}: I_i^1 \rightarrow L_1$  is a diffeomorphism.

Let  $\{J_1^{11}, J_2^{12}\}$  be the complementary intervals of  $I_1^1$  in  $J_1^1$  and  $\{J_2^{11}, J_2^{12}\}$  be the complementary intervals  $I_2^1$  in  $J_2^1$ .

Continuing with this process, in the  $k$ -th step, we obtain  $r_k = 2^k - 1$  intervals  $I_1, \dots, I_{r_k}$  such that, for each  $i \in \{1, \dots, k\}$ , there is  $n_i$  so that  $f^{n_i}: I_{r_i} \rightarrow L_1$  is a diffeomorphism.

Now, let  $\{J_1^{(k)}, \dots, J_{r_k+1}^{(k)}\}$  be the complementary intervals of  $\bigcup_{i=1}^{r_k} I_i$  in  $L_1$  and  $\mu$  be an invariant measure given by the Lemma 2.2. Then, for any interval  $I$ , there is a constant  $c$  such that

$$\mu(I) \leq c m(I) = c |I|. \quad (7)$$

Take  $\epsilon_k = \min_i \{\mu(J_i^{(k)})\} \leq c \min_i \{|J_i^{(k)}|\}$  and put  $m_k = \lfloor \frac{1}{\epsilon_k} \rfloor$  the integer part of  $\frac{1}{\epsilon_k}$ , that is,  $m_k \leq \frac{1}{\epsilon_k} < m_k + 1$ .

Next, split each intervals  $J_i^{(k)}$  in  $2^{m_k}$  intervals  $\{J_{i,j}^{(k)}: j = 1, \dots, 2^{m_k}\}$  pairwise disjoint of equal  $\mu$ -size. Then

$$\frac{1}{2^{m_k}} \geq \mu(J_{i,j}^{(k)}) = \frac{\mu(J_i^{(k)})}{2^{m_k}} \geq \frac{\epsilon_k}{2^{m_k}} > \frac{1}{2^{m_k}(m_k + 1)} \quad \text{for } j = 1, \dots, 2^{m_k}. \quad (8)$$

Consider the interval  $\left(-\frac{1}{m_k^3}, \frac{1}{m_k^3}\right)$ . Since  $\mu$  is  $f$ -invariant, inequality (7) implies that

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{4m_k} f^{-j}\left(-\frac{1}{m_k^3}, \frac{1}{m_k^3}\right)\right) &\leq \sum_{j=1}^{4m_k} \mu\left(f^{-j}\left(-\frac{1}{m_k^3}, \frac{1}{m_k^3}\right)\right) = \sum_{i=1}^{4m_k} \mu\left(-\frac{1}{m_k^3}, \frac{1}{m_k^3}\right) \\ &\leq 2c \sum_{i=1}^{4m_k} \frac{1}{m_k^3} = \frac{8c}{m_k^2}. \end{aligned} \quad (9)$$

**Claim 1.** *There is a set  $\mathcal{R}_i \subset \{1, \dots, 2^{m_k}\}$  with  $\#\mathcal{R}_i = 2^{m_k-1}$ , such that for each  $r \in \mathcal{R}_i$  there is a point  $x \in J_{i,r}^{(k)}$  such that*

$$x \notin \bigcup_{j=1}^{4m_k} f^{-j}\left(-\frac{1}{m_k^3}, \frac{1}{m_k^3}\right).$$

*Proof.* The idea of the proof is to count the number of intervals that does not satisfy this property. To do so, consider the set

$$\mathcal{R}_i^C := \left\{ j, 1 \leq j \leq 2^{m_k}; J_{i,j}^{(k)} \subset \bigcup_{j=1}^{4m_k} f^{-j}\left(-\frac{1}{m_k^3}, \frac{1}{m_k^3}\right) \right\}.$$

We want show that  $\#\mathcal{R}_i^C < 2^{m_k-1}$ . For this we proceed as follows. Put  $\#\mathcal{R}_i^C = 2^{m_k-n_k} + N_k$  with  $0 \leq N_k < 2^{m_k-n_k}$  and consider  $\underline{j}$  such that  $\mu(J_{i,\underline{j}}^{(k)}) \leq \mu(J_{i,j}^{(k)})$  for all  $j \in \mathcal{R}_i^C$ . Then, using the equations (8) and (9) we get that

$$\frac{1}{2^{m_k}(m_k+1)}(2^{m_k-n_k} + N_k) < \mu(J_{i,\underline{j}}^{(k)}) \cdot \#\mathcal{R}_i^C \leq \mu\left(\bigcup_{j \in \mathcal{R}_i^C} J_{i,j}^{(k)}\right) \leq \frac{8c}{m_k^2}.$$

Hence we have

$$\frac{1}{2^{n_k}} + \frac{N_k}{2^{m_k}} \leq 8c \cdot \frac{m_k+1}{m_k^2} \implies n_k > 1.$$

Now  $N_k < 2^{m_k-n_k}$  implies that  $\frac{N_k}{2^{m_k}} < \frac{1}{2^{n_k}}$  and as  $n_k > 1$ , we get

$$\frac{1}{2^{n_k}} + \frac{N_k}{2^{m_k}} < \frac{2}{2^{n_k}} \leq \frac{1}{2}.$$

Thus  $\#\mathcal{R}_i^C = 2^{m_k-n_k} + N_k < 2^{m_k-1}$  and this concludes the proof of claim.  $\square$

**Claim 2.** *Let  $\mathcal{R}_i$  be as above. For all  $r \in \mathcal{R}_i$  there is  $j(i, r) \in \{1, \dots, 4m_k\}$  minimal, such that*

$$|f^{j(i,r)}(J_{i,r}^{(k)})| > \frac{1}{3m_k^3}. \quad (10)$$

*Proof.* If  $|f^s(J_{i,r}^{(k)})| \geq \frac{1}{m_k^3} > \frac{1}{3m_k^3}$  for some  $s \in j = 1, \dots, 4m_k$ , we are done. Otherwise, assume that  $|f^s(J_{i,r}^{(k)})| < \frac{1}{m_k^3}$  for some  $s \in \{1, \dots, 4m_k\}$ . If  $0 \in f^s(J_{i,r}^{(k)})$ , since Claim 1 implies that there is  $x_r \in J_{i,r}^{(k)}$  such that  $x_r \notin f^{-j}\left(-\frac{1}{m_k^3}, \frac{1}{m_k^3}\right)$  for  $j = 1, \dots, 4m_k$ , we get

that  $|f^s(J_{i,r}^{(k)})| > \frac{1}{m_k^3}$ , contradicting our hypothesis. Thus  $0 \notin f^s(J_{i,r}^{(k)})$  and reasoning as in the proof of Lemma 2.1, and recalling that  $|f'| > \eta > \sqrt{2}$  we obtain

$$\frac{1}{m_k^3} \geq |f^s(J_{i,j}^{(k)})| \geq \frac{\eta^s}{2^{m_k}(m_k+1)} = \frac{2^{s/2}}{2^{m_k}(m_k+1)} = \frac{2^{s/2-m_k}}{m_k+1} \implies s/2-m_k < 0 \implies s < 2m_k.$$

If  $|f^{s+1}(J_{i,r}^{(k)})| > \frac{1}{3m_k^3}$ , then we are done. Otherwise, if  $|f^{s+1}(J_{i,r}^{(k)})| \leq \frac{1}{3m_k^3} < \frac{1}{m_k^3}$ , reasoning as before, we get that  $0 \notin f^{s+1}(J_{i,r}^{(k)})$  and by proof of Lemma 2.1 we have that  $|f^{s+1}(J_{i,r}^{(k)})| > \eta|f^s(J_{i,r}^{(k)})|$ . Again, if  $|f^{s+2}(J_{i,r}^{(k)})| > \frac{1}{3m_k^3}$ , we are done. Otherwise, if  $|f^{s+2}(J_{i,r}^{(k)})| \leq \frac{1}{3m_k^3} < \frac{1}{m_k^3}$ , and reasoning as before, we get that  $0 \notin f^{s+2}(J_{i,r}^{(k)})$  and then

$$|f^{s+2}(J_{i,r}^{(k)})| > \eta|f^{s+1}(J_{i,r}^{(k)})| > \eta^2|f^s(J_{i,r}^{(k)})|.$$

Using this argument recursively, if  $|f^{s+2m_k-1}(J_{i,r}^{(k)})| \leq \frac{1}{3m_k^3} < \frac{1}{m_k^3}$ , then  $0 \notin f^{s+2m_k-1}(J_{i,r}^{(k)})$  and it holds

$$|f^{s+2m_k}(J_{i,r}^{(k)})| > \eta|f^{s+2m_k-1}(J_{i,r}^{(k)})| > \dots > \eta^{2m_k}|f^s(J_{i,r}^{(k)})| > 2^{m_k}|J_{i,r}^{(k)}|.$$

But inequality (8) and Lemma 2.1 imply that  $|f^s(J_{i,r}^{(k)})| > \eta^s|J_{i,r}^{(k)}| > \frac{1}{2^{m_k}(m_k+1)}$ . Therefore,

$$|f^{s+2m_k}(J_{i,r}^{(k)})| > \frac{2^{m_k}}{2^{m_k}(m_k+1)} = \frac{1}{m_k+1} > \frac{1}{3m_k^3},$$

and this finishes the proof of Claim 2.  $\square$

**Claim 3.** *There is  $n_0 > 0$  such that for all  $r \in \mathcal{R}_i$ , there is an interval  $I_{i,r} \subset f^{j(i,r)}(J_{i,r}^{(k)})$  such that for some  $m_{i,r}$  the map  $f^{m_{i,r}}: I_{i,r} \rightarrow L_1$  is surjective and*

$$|I_{i,r}| > |f^{j(i,r)}(J_{i,r}^{(k)})|^{1+n_0}. \quad (11)$$

*Proof.* If for all  $n$  there is  $r_n$  such that  $f^{m_{i,r_n}}: I_{i,r_n} \rightarrow L_1$  is surjective and  $|I_{i,r_n}| \leq |f^{j(i,r_n)}(J_{i,r_n}^{(k)})|^{1+n}$ , as  $\#\mathcal{R}_i < +\infty$ , there is a sub-sequence  $n_m$ , such that  $r_{n_m} = r_0$ ,  $\forall m$ , with  $r_0 \in \mathcal{R}_i$ , and  $|I_{i,r_0}| \leq |f^{j(i,r_0)}(J_{i,r_0}^{(k)})|^{1+n_m}$ . Since  $|f^{j(i,r_0)}(J_{i,r_0}^{(k)})|^{1+n_m} \rightarrow 0$ , as  $m \rightarrow +\infty$ , this leads to a contradiction and finishes the proof of Claim 3.  $\square$

**Claim 4.** *Let  $\tilde{I}_{i,r}^{(k)} \subset J_{i,r}^{(k)}$  with  $f^{j(i,r)}(\tilde{I}_{i,r}^{(k)}) = I_{i,r}$ , where  $I_{i,r}$  is as in Claim 3. Then*

(a) *There is a constant  $K > 0$  such that*

$$|\tilde{I}_{i,r}^{(k)}| \geq K|I_{i,r}||J_{i,r}^{(k)}|. \quad (12)$$

(b) *We have  $|I_{i,r}^{(k)}| \geq \frac{1}{2^{1+o(1))m_k}}$ .*

*Proof.* First note that the Mean Value Theorem implies

$$\frac{|\tilde{I}_{i,r}^{(k)}|}{|J_{i,r}^{(k)}|} = \frac{|(f^{j(i,r)})'(y)|}{|(f^{j(i,r)})'(x)|} \cdot \frac{|I_{i,r}|}{|f^{j(i,r)}(J_{i,r}^{(k)})|} \quad \text{for some } x \in \tilde{I}_{i,r}^{(k)}; y \in J_{i,r}^{(k)}. \quad (13)$$

Thus, it is enough to bound  $\frac{|(f^{j(i,r)})'(y)|}{|(f^{j(i,r)})'(x)|}$  to conclude the proof of item (a). To do so we proceed as follows.

As  $j(i, r)$  is minimal satisfying (10) we get

$$|f^s(J_{i,r}^{(k)})| < \frac{1}{3m_k^3} \quad \text{for } s = 1, \dots, j(i, r) - 1. \quad (14)$$

This implies, reasoning as in the proof of Claim 2, that  $0 \notin f^s(J_{i,r}^{(k)})$  for  $s = 1, \dots, j(i, r) - 1$  and hence  $f^s|_{J_{i,r}^{(k)}}$  is a diffeomorphism for  $s = 1, \dots, j(i, r) - 1$ .

Observe that by Claim 1, for each  $s \in \{1, \dots, j(i, r) - 1\}$ , there is  $x_s \in J_{i,r}^{(k)}$  such that  $f^s(x_s) \notin (-\frac{1}{m_k^3}, \frac{1}{m_k^3})$ , and so, if  $d(\cdot, \cdot)$  is the distance between sets, by equation (14) we concluded that

$$d(f^s(J_{i,r}^{(k)}), \{0\}) > \frac{1}{2m_k^3}. \quad (15)$$

Now we have

$$\begin{aligned} \left| \log \frac{(f^{j(i,r)})'(y)}{(f^{j(i,r)})'(x)} \right| &= \left| \sum_{s=1}^{j(i,r)-1} \log(f'(f^s(y))) - \log(f'(f^s(x))) \right| \\ &\leq \sum_{s=1}^{j(i,r)-1} |\log(f'(f^s(y))) - \log(f'(f^s(x)))| \\ &\leq_{by \text{ MVT}} \sum_{s=1}^{j(i,r)-1} \frac{|f''(f^s(z_s))|}{|f'(f^s(z_s))|} |f^s(y) - f^s(x)|, \quad \text{for some } z_s \in J_{i,r}^{(k)}, \\ &\leq_{by (6)} \sum_{s=1}^{j(i,r)-1} C \cdot C_1 \cdot \frac{1}{|f^s(z_s)|} \cdot |J_{i,r}^{(k)}|, \quad \text{where } C, C_1 \text{ depend only on } f \\ &\leq_{by (14)} \sum_{s=1}^{j(i,r)-1} C \cdot C_1 \cdot \frac{1}{|f^s(z_s)|} \cdot \frac{1}{3m_k^3}. \end{aligned} \quad (16)$$

Recall that equation (6) implies that  $\frac{|f''(x)|}{|f'(x)|} \leq \frac{C \cdot C_1}{|x|}$ , with  $C_1, C$  depending only of  $f$ . Thus, since  $f^s|_{J_{i,r}^{(k)}}$  is a diffeomorphism for each  $s \in \{1, \dots, j(i, r) - 1\}$ , and satisfies Property (f3) (see subsection 2.4), we get

$$|f^{j(i,r)-1}(J_{i,r}^{(k)})| > \sqrt{2} |f^{j(i,r)-2}(J_{i,r}^{(k)})| > \sqrt{2}^2 |f^{j(i,r)-3}(J_{i,r}^{(k)})| > \dots > \sqrt{2}^s |f^{j(i,r)-(s+1)}(J_{i,r}^{(k)})|.$$

Using this inequality together with (14) and (15) and replacing in the last term of equation (16) we get that

$$\left| \log \frac{(f^{j(i,r)})'(y)}{(f^{j(i,r)})'(x)} \right| \leq C \cdot C_1 \sum_{s=1}^{j(i,r)-1} 2m_k^3 \cdot \frac{1}{3m_k^3} \left( \frac{1}{\sqrt{2}} \right)^{j(i,r)-s-1} < \frac{2}{3} \cdot C \cdot C_1 \cdot \sqrt{2}(\sqrt{2} + 1).$$

Setting  $K := e^{-\frac{2}{3} \cdot C \cdot C_1 \cdot \sqrt{2} \cdot (\sqrt{2}+1)}$ , we bound  $\frac{|(f^{j(i,r)})'(y)|}{|(f^{j(i,r)})'(x)|}$  and inequality (13) follows, implying that inequality (12) holds. The proof of Claim 4(a) is finished.

To prove item (b), note that equations (10), (11) and (12) imply that

$$|\tilde{I}_{i,r}^{(k)}| \geq K \left( \frac{1}{3m_k^3} \right)^{1+n_0} |J_{i,r}^{(k)}|.$$

Now, replacing  $B = \log K - (1 + n_0) \log 3$  in inequality (8) we get

$$\frac{B - 3(1 + n_0) \log m_k}{-\ln(m_k + 1) - m_k \log 2} \leq \frac{\log \left( K \left( \frac{1}{3m_k^3} \right)^{1+n_0} \right)}{\log |J_{i,r}^{(k)}|} \leq \frac{B - 3(1 + n_0) \log m_k}{-m_k \log 2}.$$

Hence,  $\lim_{k \rightarrow \infty} \frac{\log \left( K \left( \frac{1}{3m_k^3} \right)^{1+n_0} \right)}{\log |J_{i,r}^{(k)}|} = 0 \implies K \cdot \left( \frac{1}{3m_k^3} \right)^{1+n_0} = |J_{i,j}^{(k)}|^{o(1)}$ , and combining with (3) we obtain

$$|\tilde{I}_{i,r}^{(k)}| \geq |J_{i,r}^{(k)}|^{1+o(1)} = \frac{1}{2^{(1+o(1))(1+o(1))m_k}} = \frac{1}{2^{(1+o(1))m_k}},$$

since (8) implies that  $\lim_{k \rightarrow +\infty} \frac{\log |J_{i,r}^{(k)}|}{\log 2^{-m_k}} = 1$  which is equivalent to write  $|J_{i,r}^{(k)}| = 2^{-(1+o(1))m_k}$ . The proof of item (b) is complete.  $\square$

The next step is to construct the regular Cantor with Hausdorff dimension close to 1. For this, consider the collection of surjective maps

$$\{g_{i,r} = f^{m_{i,r}} \circ f^{j(i,r)} : \tilde{I}_{i,r}^{(k)} \rightarrow L_1; r \in \mathcal{R}_i\}.$$

Let  $g_{ki} : L_k^i = \bigcup_{r \in \mathcal{R}_i} \tilde{I}_{i,r}^{(k)} \rightarrow L_1$  be defined by  $g_{ki} = g_{i,r}|_{\tilde{I}_{i,r}^{(k)}}$  and  $C_k^i$  be the Cantor set defined by the intervals  $\tilde{I}_{i,r}^{(k)}$  and  $g_{i,r}$ , i.e.,

$$C_k^i = \bigcap_{n \geq 1} g_{ki}^{-n}(L_k^i).$$

The final step is to show that  $HD(C_k^i) \rightarrow 1$  as  $k \rightarrow +\infty$ . For this, recall that the Hausdorff dimension of  $C_k^i$  is the exponent  $d$  satisfying

$$\sum_r |\tilde{I}_{i,r}^{(k)}|^d = 1.$$

Therefore, since  $\#\mathcal{R}_i = 2^{m_k-1}$  and (3) holds, setting  $d = HD(C_k^i)$  we get

$$2^{m_k-1} \cdot \left( \frac{1}{2^{(1+o(1)) \cdot m_k}} \right)^d \leq 1 = \sum_{r=1}^{2^{m_k-1}} |\tilde{I}_{i,r}^{(k)}|^d.$$

Hence

$$\begin{aligned} (m_k - 1) \log 2 &\leq (1 + o(1)) \cdot m_k \cdot d \cdot \log 2 \implies \\ 1 - o(1) = \frac{m_k - 1}{m_k} &\leq (1 + o(1)) \cdot d \implies \\ 1 - o(1) &\leq (1 + o(1)) \cdot d. \end{aligned}$$

Thus,  $1 - o(1) \leq d \leq 1$ , finishing the proof of Theorem 1.  $\square$

## 4 Proof of Theorem B

Here we prove Theorem B. For this, let  $\Gamma = \{(x, y, 1) : x = 0\}$ , recall (2), and

$$\Lambda_P = \overline{\bigcap_{i \geq 1} P^i(S \setminus \Gamma)}, \quad \text{be as in equation (4).}$$

For each  $k > 0$ , let  $C_k$  be the regular Cantor set given by Theorem 1 and define

$$\Lambda_P^k = \{(x, y) : x \in C_k\}. \quad (17)$$

By definition of  $P$ , for each  $k$ ,  $\Lambda_P^k$  is hyperbolic for  $P$ . Moreover,  $\Lambda_P^k \subset \Lambda_P^{k+1}$  and

$$H(\Lambda_P^k) = HD({}_u K_P^k) + HD({}_s K_P^k) = HD(C_k) + HD({}_s K_P^k),$$

where  ${}_s K_P^k$  and  ${}_u K_P^k = C_k$  are the stable and unstable Cantor sets associated to the transitive hyperbolic set  $\Lambda_P^k$  (cf. [PT93]). As  ${}_s K_P^k$  is a regular Cantor set, there is  $\xi > 0$  such that  $HD({}_s K_P^1) > \xi$ . Hence

$$H(\Lambda_P^k) = HD(C_k) + HD({}_s K_P^k) \geq HD(C_k) + HD({}_s K_P^1) > HD(C_k) + \xi.$$

Thus, Theorem 1 implies that  $H(\Lambda_P^k) > 1$  for  $k$  large enough. Since  $\Lambda_P^k \subset \Lambda_P$ , this finishes the proof of Theorem B.

**Proof of Theorem A.** Note that the geometric Lorenz attractor  $\Lambda$  satisfies

$$\Lambda = \left( \bigcup_{t \in \mathbb{R}} X^t(\Lambda_P) \right) \cup O, \quad \text{where } O \text{ is the singularity.}$$

Thus,

$$HD(\Lambda) \geq 1 + HD(\Lambda_P) > 2. \quad \square$$

## 5 Lagrange and Markov Spectra

In this section we prove Theorem C. For this, we first prove at Lemma 5.1 that small perturbations of the Poincaré map  $P$  restrict to  $\Lambda_P^k$ ,  $\Lambda_P^k$  defined at (17), can be realized as Poincaré maps of small perturbations of the initial geometric Lorenz flow  $X^t$ . Thus, taking  $k$  such that  $H(\Lambda_P^k) > 1$ , we recover the properties described in [RM15a] needed to apply [RM15a, Main Theorem], obtaining non empty interior in the Lagrange and Markov spectrum.



## 5.1 Perturbations of Poincaré Map

Fixed  $k$  with  $HD(\Lambda_P^k) > 1$ . By construction, there is  $\epsilon > 0$  small so that  $d(\Lambda_P^k, \Gamma) > 2\epsilon$ , where  $\Gamma = \{(x, y, 1) : x = 0\}$ . Let  $\mathcal{U}_P$  be a  $C^2$  neighborhood of  $P$  such that, if  $\tilde{P} \in \mathcal{U}_P$  and  $\Lambda_{\tilde{P}}^k$  is the hyperbolic continuation of  $\Lambda_P^k$ , then  $d(\Lambda_{\tilde{P}}^k, \Gamma) > \epsilon$ .

The next Lemma states that in a neighborhood of  $\Lambda_{\tilde{P}}^k$ , we can recover  $\tilde{P} \in \mathcal{U}_P$  as a Poincaré map associated to a geometric Lorenz flow  $\tilde{X}^t$ ,  $C^2$ -close to  $X^t$ .

**Lemma 5.1.** *Given  $\tilde{P} \in \mathcal{U}_P$  there is a geometric Lorenz flow  $\tilde{X}^t$ ,  $C^2$ -close to  $X^t$ , such that the restriction to  $\Lambda_{\tilde{P}}^k$  of the Poincaré map associated to  $\tilde{X}^t$  coincides with the restriction of  $\tilde{P}$  to  $\Lambda_{\tilde{P}}^k$ .*

*Proof.* For the proof we construct explicitly a flow  $\tilde{X}^t$ , with the desired properties. For this, we proceed as follows.

Let  $\tilde{\mathcal{R}} = R_1 \cup R_2 \cup \dots \cup R_m$  be a Markov partition of  $\Lambda_{\tilde{P}}^k$  and let  $U_i \subset S$  be an open set with  $R_i \subset U_i$ ,  $d(U_i, \Gamma) > \frac{\epsilon}{2}$  for all  $i$ , and such that if  $\tilde{P}(x, y) \in R_i$ , then  $P(x, y) \in U_i$ . The tubular flow theorem applied to  $X$ , give local charts  $\psi_i : U_i \times [-1, 1] \rightarrow \mathbb{R}^3$  for  $i \in \{1, \dots, m\}$  satisfying

$$\psi_i(U_i \times \{0\}) \subset S \quad \text{and} \quad D(\psi_i)_{(x,y,t)}(0, 0, 1) = X(\psi_i(x, y, t)). \quad (18)$$

Put  $W_i := \psi_i(U_i \times (-1, 1))$ . Without loss of generality, we can assume that

$$W_i \cap W_j = \emptyset \quad \text{if} \quad i \neq j.$$

We denote by  $\tilde{P}_i$  and  $P_i$  the map  $\tilde{P}$  and  $P$  in these coordinates.

Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^\infty$ -function bump function such that  $\varphi(t) = 0$  for  $t \leq -1$  and  $\varphi(t) = 1$  for  $t \geq 1$ . Define the following flow on  $U_i \times [-1, 1]$ :

$$\phi_i^t(x, y, 0) = (P_i(x, y) + \varphi(3t + 1)(\tilde{P}_i(x, y) - P(x, y)), t).$$

Note that

$$\phi_i^t(x, y, 0) = \begin{cases} (P_i(x, y), t), & \text{if } t \leq -\frac{2}{3} \\ (\tilde{P}_i(x, y), t), & \text{if } t \geq 0 \end{cases}.$$

Consider the vector field on  $U_i \times [-1, 1]$  given by

$$\phi_i(\phi_i^t(x, y, 0)) = \frac{\partial}{\partial t} \phi_i^t(x, y, 0) = (3\varphi'(3t + 1)(\tilde{P}_i(x, y) - P(x, y)), 1).$$

By the equation (5.1), this vector field satisfies

$$\phi_i(\phi_i^t(x, y, 0)) = \begin{cases} (0, 0, 1), & \text{if } t \leq -\frac{2}{3} \\ (0, 0, 1), & \text{if } t \geq 0 \end{cases}. \quad (19)$$

Let the vector field on  $W_i = \psi_i(U_i \times (-1, 1))$  defined by

$$Y_i(\psi_i(x, y, t)) = D(\psi_i)_{(x,y,t)}(\phi_i(\phi_i^t(x, y, 0))).$$

By equations (18) and (19) it holds that

$$Y_i(\psi_i(x, y, t)) = X(\psi_i(x, y, t)) \quad \text{for } t \leq -\frac{2}{3} \quad \text{and} \quad t \geq 0.$$

Let  $\mathcal{W}$  be the open set  $\mathcal{W} = \bigcup_{i=1}^m \psi_i(U_i \times (-1, 1)) = \bigcup_{i=1}^m W_i$  and consider the vector field  $Y : \mathcal{W} \rightarrow \mathbb{R}^3$  given by  $Y = Y_i|_{W_i}$ . Finally, define the vector field  $\tilde{X}$  by

$$\tilde{X} := \begin{cases} Y, & \text{on } \mathcal{W} \\ X, & \text{outside of } \mathcal{W} \end{cases}.$$

Since  $\tilde{P} \in \mathcal{U}_P$ , equation (5.1) implies that  $\tilde{X}$  is  $C^2$ -close to  $X$ . If  $\tilde{X}^t$  is the flow associated to the vector field  $\tilde{X}$ , equations (5.1) and (5.1) imply that the Poincaré map associated to  $\tilde{Y}^t$  restrict to  $\Lambda_P^k$  is equal to  $\tilde{P}$  restrict to  $\Lambda_P^k$ . To finish the proof, note that  $d(U_i, \Gamma) > \frac{\epsilon}{2}$  for all  $i$ , and thus,  $\tilde{X}^t$  is a geometric Lorenz flow, as desired.  $\square$

Let  $U$  be the open set where the flow acts and  $\Lambda \subset U$ , where  $\Lambda$  is the initial Lorenz attractor (cf. Section 2.2). Given  $F \in C^0(U, \mathbb{R})$ , define the function  $\max F_X : S^* \rightarrow \mathbb{R}$  by

$$\max F_X(x) := \max_{t_-(x) \leq t \leq t_+(x)} F(X^t(x))$$

where  $t_-(x), t_+(x)$  are such that  $P^{-1}(x) = X^{t_-(x)}(x)$  and  $P(x) = X^{t_+(x)}(x)$ .

**Remark 1.** Note that this definition depends on the flow  $X^t$ , or equivalently on the vector field  $X$ . Also,  $\max F_X(x)$  is the maximum value of  $F$  along the segment of orbit of  $X^t(x)$ ,  $t_-(x) \leq t \leq t_+(x)$ . The map  $\max F_X$  is always continuous, but even if  $F$  is  $C^\infty$ ,  $\max F_X$  can be only a continuous function. The next lemma means that we can construct a new Markov partition from the initial one, in such away the sub-horseshoes associated to the new Markov partition have Hausdorff dimension close to the initial horseshoes and the map  $\max F_X(x)$  is  $C^1$  in this new partition.

**Lemma 5.2.** Let  $\Lambda_P^k$  be defined by equation (17). Then there is a  $C^2$ -open and dense set  $\mathcal{B}_X \subset C^\infty(U, \mathbb{R})$  such that given  $\beta > 0$ , for any  $F \in \mathcal{B}_X$  there are sub-horseshoes  ${}_k\Lambda_F^{s,u} \subset \Lambda_P^k$  with  $HD({}_kK_F^s) \geq HD(K_k^s) - \beta$ ,  $HD({}_kK_F^u) \geq HD(K_k^u) - \beta$  and a Markov partition  ${}_kR_F^{s,u}$  of  ${}_k\Lambda_F^{s,u}$ , such that the map  $\max F_X|_{S \cap {}_kR_F^{s,u}} \in C^1(S^* \cap {}_kR_F^{s,u}, \mathbb{R})$ , where  ${}_kK_F^{s,u}$ ,  $K_k^{s,u}$  are regular Cantor sets that describe the transverse geometry of the unstable/stable leaves of  $W^{u,s}({}_k\Lambda_F^{s,u})$  and  $W^{u,s}(\Lambda_P^k)$  respectively.

The proof of Lemma 5.2 can be found in [RM15b, Lemma 18, pp 35].

**Remark 2.** Let  $x \in \text{int}(S^*)$ , recall (2), such that  $P(x) = X^{t_+(x)}(x) \in \text{int}(S^*)$ . The Tubular Flow Box Theorem implies that there is a neighborhood  $U_x \subset S^*$  of  $x$ , a diffeomorphism  $\varphi : U_x \times (-\epsilon, t_+(x) + \epsilon) \rightarrow \varphi(U_x \times (-\epsilon, t_+(x) + \epsilon)) \subset U$  such that  $D\varphi_{(z,t)}(0, 0, 1) = X(\varphi(z, t))$  for every  $(z, t) \in U_x \times (-\epsilon, t_+(x) + \epsilon)$ . Moreover, as the elements of the Markov partition are disjoint, have small diameter and  $\Lambda_P^k$  is compact, we can suppose, without loss, that there is a finite number of open sets  $U_{x_i}$  such that  $U_{x_i} \cap U_{x_j} = \emptyset$  and  $\Lambda_P^k \subset \bigcup U_{x_i}$  for certain  $x_i \in \Lambda_P^k$ . Denote  $\varphi_i : U_{x_i} \times (-\epsilon, t_+(x_i) + \epsilon) \rightarrow \varphi_i(U_{x_i} \times (-\epsilon, t_+(x_i) + \epsilon)) \subset U$  such that  $(D\varphi_i)_{(z,t)}(0, 0, 1) = X(\varphi_i(z, t))$ .

Keeping the notation of the previous Lemma we have:

**Corollary 1.** The above property (existence of  $\mathcal{B}_X \subset C^\infty(U, \mathbb{R})$  satisfying Lemma 5.2) is robust in the following sense: If  $Y$  is a vector field  $C^1$ -close to  $X$ , then  $\mathcal{B}_Y = \mathcal{B}_X$  and for any  $F \in \mathcal{B}_Y$ , it holds that  $\max F_Y \in C^1(S^* \cap {}_kR_F^{s,u}, \mathbb{R})$ .

### **Proof of Theorem C.**

Let  $k$  be large enough such that the transitive horseshoe  $\Lambda_P^k$  (in the proof of Theorem B) has Hausdorff dimension greater than 1. Let  $F \in B_X$ ,  ${}_k\Lambda_F^{s,u}$  and  ${}_kK_F^{s,u}$  as in Lemma 5.2 with

$$HD({}_kK_F^{s,u}) \geq HD(K_k^{s,u}) - \beta, \quad \text{for some } \beta > 0.$$

We can assume that  $\beta$  is small enough in order to have  $HD({}_k\Lambda_F^s \cup {}_k\Lambda_F^u) > 1$ .

Define the sub-horseshoe of  $\Lambda_P^k$  by  $\Lambda_F := {}_k\Lambda_F^s \cup {}_k\Lambda_F^u$  and put  $R_F = {}_kR_F^s \cup {}_kR_F^u$ . Now, consider the pair  $(P, \Lambda_F)$  with  $HD(\Lambda_F) > 1$ , then by [RM15a, Main Theorem], there exist  $\tilde{P} \in \mathcal{U}_P$ , ( $\mathcal{U}_P$  as Lemma 5.1) such that if  $\tilde{\Lambda}_F$  is the hyperbolic continuation of  $\Lambda_F$ , then the pair  $(\tilde{P}, \tilde{\Lambda}_F)$  satisfies the conclusions of [RM15a, Main Theorem], *i.e.*, if

$$M_f(\tilde{\Lambda}_F) = \{x \in \tilde{\Lambda}_F : f(x) \geq f(y) \text{ for all } y \in \Lambda_F^w\}$$

is set of maximum points of  $f$  in  $\tilde{\Lambda}_F$  and  $e_z^{s,u}$  are unit vectors in the stable and unstable bundle of  $\tilde{\Lambda}_F$ , respectively, then the set  $\mathcal{H}_1(\tilde{P}, \tilde{\Lambda}_F)$  defined below is open and dense

$$\mathcal{H}_1(\tilde{P}, \tilde{\Lambda}_F) = \{f \in C^1(S^* \cap R_F, \mathbb{R}) : \#M_f(\tilde{\Lambda}_F) = 1, \text{ for } z \in M_f(\tilde{\Lambda}_F), (Df)_z(e_z^{s,u}) \neq 0\}.$$

Note that  $\mathcal{H}_1(\tilde{P}, \tilde{\Lambda}_F)$  is the set of functions  $f$  such that if  $z \in M_f(\tilde{\Lambda}_F)$ , then the gradient  $\nabla(f(z))$  is not collinear to the stable direction neither to the unstable direction at  $z$ . Therefore, by [RM15a, Main Theorem], for every  $f \in \mathcal{H}_1(\tilde{P}, \tilde{\Lambda}_F)$  the Lagrange and Markov Dynamical Spectrum has non empty interior, *i.e.*,

$$\text{int } M(f, \tilde{\Lambda}_F) \neq \emptyset \quad \text{and} \quad \text{int } L(f, \tilde{\Lambda}_F) \neq \emptyset. \quad (20)$$

Next we use the maps given by Corollary 1 to recover the same property for the spectrum of some perturbation of  $F$ , and so, the same property for perturbations of the flow, recall definition of  $F$  above. For this we proceed as follows.

By Corollary 1, the function  $\max F_{\tilde{X}}|_{S^* \cap R_F}$  is  $C^1$  ( $\tilde{X}$  given by Lemma 5.1) and since the set  $\mathcal{H}_1(\tilde{P}, \tilde{\Lambda}_F)$  is open and dense, using local coordinates as in Remark 2 respect to the flow  $\tilde{X}^t$ , we can find  $g \in C^2(S^*, \mathbb{R})$  such that

$$\max F_{\tilde{X}}|_{S^* \cap R_F}(x_1, x_2, x_3) + g(x_1, x_2) \in \mathcal{H}_1(\tilde{P}, \tilde{\Lambda}_F). \quad (21)$$

Observe that, by Lemma 5.1 the map associated to  $\tilde{X}^t$  restrict to  $\tilde{\Lambda}_F$  is equal to  $\tilde{P}$  restrict to  $\tilde{\Lambda}_F$ .

Put  $H(x_1, x_2, x_3) = F(x_1, x_2, x_3) + g(x_1, x_2)$ . Since  $g$  does not depend on coordinate  $x_3$ , we have that  $\max H_{\tilde{X}}|_{S^* \cap R_F} = \max F_{\tilde{X}}|_{S^* \cap R_F} + g$  and hence, by (21), we get  $\max H_{\tilde{X}}|_{S^* \cap R_F} \in \mathcal{H}_1(\tilde{P}, \tilde{\Lambda}_F)$ .

Therefore, since  $M(\max H_{\tilde{X}}, \tilde{\Lambda}_F) = \left\{ \sup_{n \in \mathbb{R}} \max H_{\tilde{X}}(\tilde{P}^n(x)) : x \in \tilde{\Lambda}_F \right\} \subset M(H, \tilde{X})$ , equation (20) implies that

$$\text{int } M(H, \tilde{X}) \neq \emptyset.$$

Analogously,  $L(\max H_{\tilde{X}}, \tilde{\Lambda}_F) \subset L(H, \tilde{X})$ , therefore  $\text{int } L(H, \tilde{X}) \neq \emptyset$ . All together finishes the proof of Theorem C.  $\square$

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